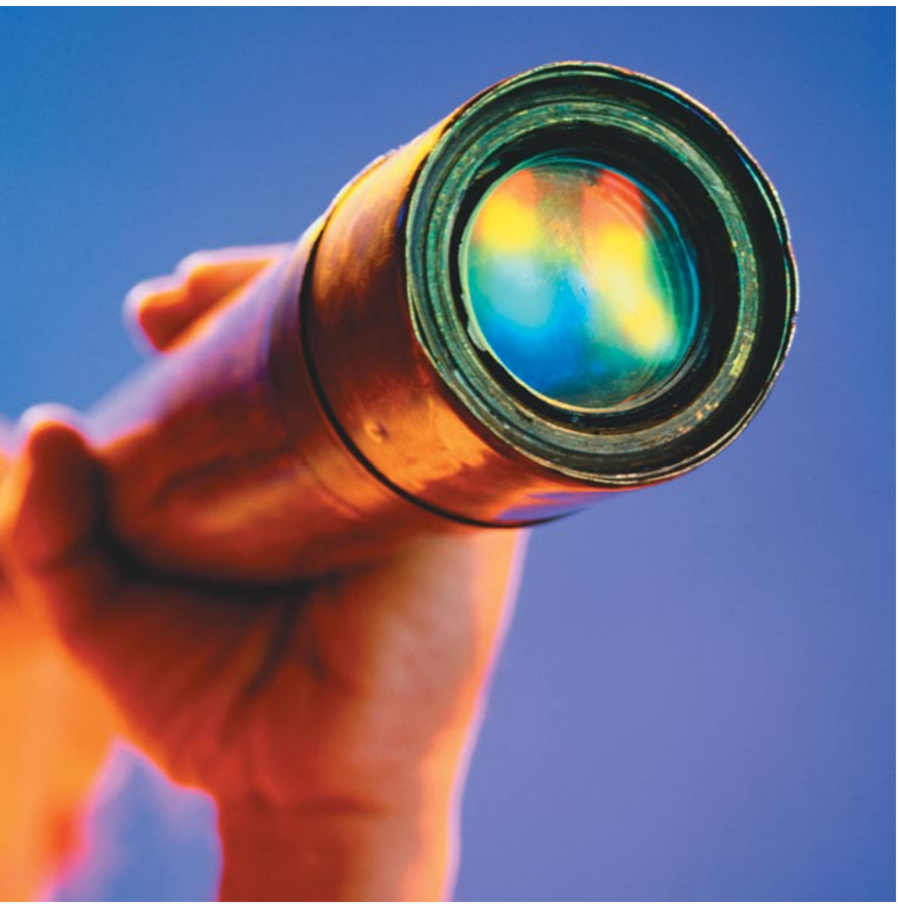
Teaching Algebra and Geometry Concepts by Modeling Telescope Optics

Lauren M. Siegel, Gail Dickinson, Eric J. Hooper, and Mark Daniels



The vision statement contained in NCTM's *Principles and Standards for School Mathematics* (2000) calls for teachers to provide a mathematically rich curriculum, draw on knowledge from a wide variety of mathematical topics, use reasoning and proof techniques to confirm or disprove conjectures, work with students individually and in groups, and engage in purposeful use of technology. In the UTeach Program in the College of Natural Sciences, University of Texas, Austin, preservice teachers develop curricula that contextualize mathematics and offer rich and extended problems through the Project-Based Instruction course.

The lessons presented here were produced by author Lauren Siegel, a member of a team of preservice teachers who conducted a collaborative field experience to deliver project lessons in astronomical telescope design and construction to students in second-year algebra courses. This sequence of investigations explores how and why the physical and mathematical properties of parabolic mirrors both enable and constrain our ability to build and use telescopes to focus light from distant objects. The mirrors used in these Newtonian telescopes are paraboloids. The geometry of conic sections typically studied in algebra 2 and precalculus courses is directly related to the function of the mirrors and thus provides a strong link between this material and the state's content requirements for these courses. This sequence of investigations, inspired by existing lessons in NCTM publications, mathematics textbooks, lecture notes, and colleagues' suggestions and contributions, were refined and enhanced on the basis of student responses during two separate presentations of the entire project by the same team of UTeach preservice teachers. Various approaches—including generating and exploring computer models, exploring traditional proofs and even making paper models—are woven together into a coherent set of eleven investigations for use in mathematics classrooms.

Preparation for the preservice teachers included lessons in astronomy, optics, and telescope function by author Eric Hooper, whose training enabled the inclusion of real design constraints and optical properties into the field experience. Teachers without this background can still use the investigations presented here as stand-alone units in a mathematics class. Alternatively, they can coordinate with instructors in physics, astronomy, or the history of science in a manner recalling the cross-disciplinary work of Ryden (1999).

The activities described here, which followed a lab activity on light reflection and preceded construction of the telescope, were presented to students at a Central Texas high school in approximately 1.5 hours. In a regular classroom setting, this ensemble of investigations will likely take a minimum of three class periods and more likely a week, depending in part on whether related homework is assigned. The activities naturally divide into three sections: understanding parabolas through a variety of approaches (investigations 1–5); connecting the mathematical properties of parabolas with light reflection by parabolic mirrors (investigations 6 and 7); and more specific modeling of real telescopes like the ones built during the field experience (investigations 8–11). Grouping the investigations this way allows teachers with less time to choose the parts most relevant to their curriculum. Teachers who wish to expand these investigations can guide students to develop and discuss their own solutions.

The investigations include activities and guided questions with answers suitable for individual work, small-group interactions, or whole-class discussions. Answers include some student responses from classes taught in 2005. They also provide anticipated responses and guidance for additional instruction. The format encourages students and teachers to explore the topic through discourse; to find answers through Euclidean proof, modeling, experimentation, and investigation; and to connect abstract ideas with concrete experiences.

UNDERSTANDING PARABOLAS

Investigation 1: Making paper-folding models and defining a parabola

Students make parabolas by drawing a line and a point not on the line (the *focus*) on wax paper and then repeatedly "folding" the point onto different parts of the line. Results are a smooth section and a creased section. Drawing along the borders, students can generate parabolas (see NCTM's applet on parabola paper folding: my.nctm.org/eresources/view_article.asp?add=Y&article_id=2074&page=9#).

Questions

1. What factors influence the shape of the parabola?
2. Why does this folding procedure make a parabola?
3. Choose a point on your parabola. Draw a line from it to your focus and also drop a perpendicular to the line. Redo the fold that generated that point. What do you notice? Conduct the same procedure on a second point for verification.

Answers

1. The distance from the point to the line determines the final shape of the parabola.
2. Students responded with "[S]omething to do with angles?" The inclusion of a Euclidean proof in investigation 3 was added to answer this question.
3. Points on the parabola are equidistant from the point and the line. A parabola is a collection of points equidistant from the focus (a point) and the directrix (a line).

Investigation 2: Building a Geometer's Sketchpad model of the paper parabolas

Students use Geometer's Sketchpad (GSP) or similar software to construct **figure 1**. Using the command Display and then Trace enables students to make a tracing on the line (the perpendicular bisector of

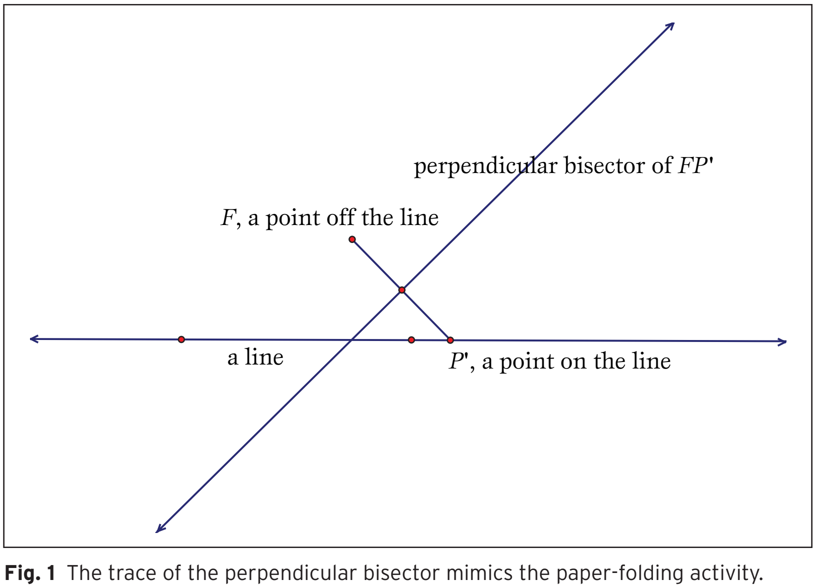


Fig. 1 The trace of the perpendicular bisector mimics the paper-folding activity.

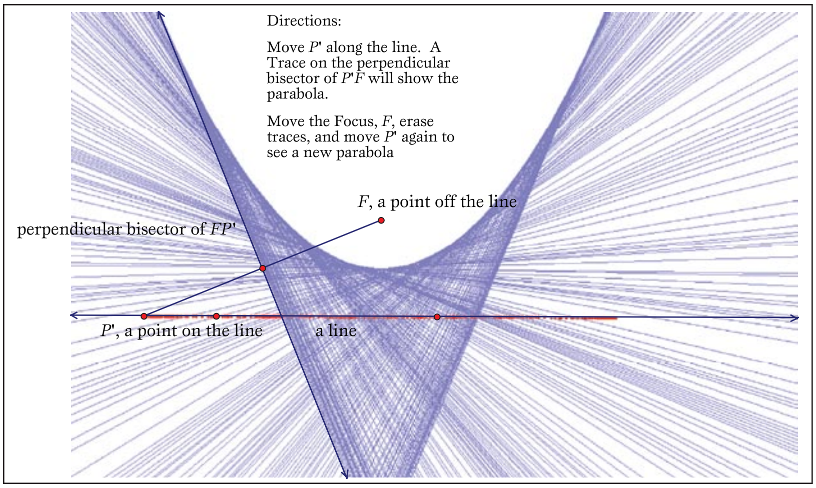


Fig. 2 Results of tracing the perpendicular bisector as P′ slides along the line

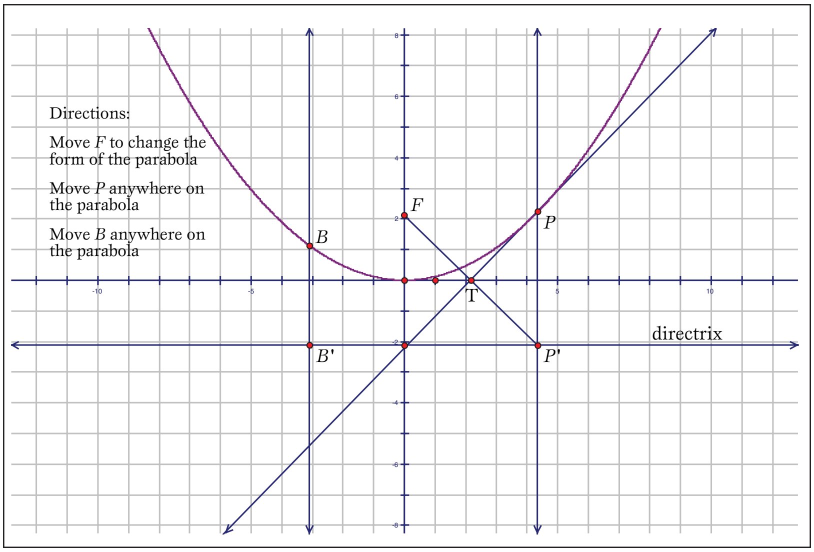


Fig. 3 A GSP sketch showing that perpendicular bisectors are tangents to the parabola

$\overline{FP'}$) that corresponds to the crease from the fold. Drag point P' along the line and note any patterns (**fig. 2**).

Questions

1. How is the distance of the focus from the line related to the shape of the parabola?
2. How does this distance relate to the focal length, that is, the distance from the focus to the vertex?
3. Is your wax-paper parabola consistent with the model in GSP?

Answers

1. The parabola is wide when the focus is far from the line and very narrow when the focus is close to the line.
2. Focal length is half the distance from the focus to the line.
3. Yes, students can change the model on the screen. Holding the wax-paper models to the screen to see the match was particularly effective for our students.

Investigation 3: Proving that the paper folds are tangents to a parabola

Students can now define a parabola as the set of points equidistant from a point and a line. Consider a point, P, somewhere on the parabola. Let the focus be F, and let P' be the point on the directrix closest to P and therefore vertically below P. See **fig. 3** (adapted from a proof at www.cut-the-knot.org/ctk/Parabola.shtml by Alex Bogomolny; a similar argument appears in Dennis and Confrey [1995]).

Questions

1. What kind of triangle is FPP'?
2. If T is the midpoint of $\overline{FP'}$, what do we know about $\overline{TP}$?
3. Let B be another point on the parabola and represent the point on the directrix, B', that is directly below B. How does the length of $\overline{BP'}$ compare with that of $\overline{BB'}$ and $\overline{BF}$?
4. What can we conclude?

Answers

1. FPP' is an isosceles triangle because it has two congruent sides, FP and $\overline{PP'}$, according to the definition of a parabola.
2. $\overline{TP}$ is a perpendicular bisector of $\overline{FP'}$.
3. $BP' > BB'$ and so $BP' > BF$.
4. All points on the parabola except P are closer to F than they are to P'. That is, all points on the parabola are on one side of $\overline{TP}$ except point P. $\overline{TP}$ is therefore tangent to the curve at P.

Investigation 4: Placing the paper parabolas in the Cartesian plane

We will be placing wax-paper parabolas in the Cartesian plane so that we can develop a general formula. Give the students grid paper and have them place and tape their wax-paper figures onto a coordinate grid.

Questions

1. Develop and describe a reasoned approach to this task so that the placement will facilitate finding a formula.
2. Place your wax-paper parabola onto graph paper so that the directrix is parallel to the x-axis, the focus is on the y-axis, and the curve faces upward with its vertex at (0, 0). Can all our parabolas be placed this way? Will this orientation limit our proof in such a way that the formula we generate will work only for some parabolas?
3. Use a ruler to measure the distance from the focus to the origin and from the directrix to the origin. What do you find? Is your finding true for each student's figure?

Answers

1. Students gave varying partial answers. Discuss until the class arrives at the directions for the sec-

ond group of investigations (Connecting Parabolas with Light Reflection).

2. The parabolas were made without regard to orientation of the line and the point, so all can be oriented as upward facing with vertex at the origin and focus on the y-axis. The resulting general formula applies to upward-facing parabolas only. For a GSP demonstration, use the Graph menu and then Show Grid to overlay a grid. In our classes, we used a GSP image projected onto a large sheet of poster paper so that hand-drawn axes, coordinates, and notes could be added. GSP can also be used to display the coordinate plane (**fig. 3**).

3. The focus, F, and the directrix are equidistant from the origin within measurement uncertainties.

Investigation 5: Using Cartesian coordinates to find an algebraic formula for a parabola

Choose an arbitrary point, P, with coordinates (x, y) on the parabola used in the previous investigation (**fig. 3**).

Questions

1. What can we say about the coordinates of the focus?
2. What are the coordinates of the point on the directrix, P', which is directly beneath P?
3. How can we express the distances from F to P and from P' to P in terms of their coordinates?
4. Use the distance formula to write $FP = P'P$ in terms of the coordinates of the points.
5. The formula we derived is for an upward-facing parabola. Can you derive the formulas for three other parabolas: downward, right facing, and left facing? (See Sullivan and Sullivan 1998.)

Answers

1. The focus is placed on the y-axis so the x coordinate is 0. The y coordinate is a constant because the focus does not change. Use $(0, a)$ for the coordinates of F. Thus, a is the focal length.
2. P' will have the coordinates $(x, -a)$.
3. We can use the distance formula:

$$\left|FP\right| = \sqrt{(0-x)^2 + (a-y)^2}.$$

4. $|FP| = |PP'|$, so

$$\sqrt{(-x)^2 + (a-y)^2} = \sqrt{(y+a)^2}.$$

Now square both sides. Because the terms under the square root symbols are all positive, there is no lost solution. We have

$$x^2 + a^2 - 2ay + y^2 = y^2 + 2ay + a^2$$

and then

$$x^2 = 4ay$$

and

$$y = x^2/4a.$$

This is the general form for an upward-facing parabola with its vertex at $(0, 0)$ and focus at $(0, a)$.

5. Challenge students to find the remaining three formulas.

CONNECTING PARABOLAS WITH LIGHT REFLECTION

At this point, the lessons move from a general discussion of parabolas (investigations 1–5) to drawing specific connections with the reflection of light in parabolic mirrors (investigations 6 and 7). Students who have not studied light reflection as ours did may need a brief introduction or refresher so that they understand that the angle of incidence is the same as the angle of reflection with respect to the normal for light hitting an arbitrarily curved mirrored surface.

Investigation 6: Showing that incident light parallel to the central axis reflects to the focus for parabolic mirrors

In this activity, the students use GSP to show the reflecting property of the parabola.

Questions

1. What happens to a ray of light when it hits a flat mirror? A concave mirror?
2. When a ray of light hits a parabolic mirror, how is it reflected?
3. In GSP, build a file that models the path of reflection of light hitting a parabola and that allows you to set the angle of incident light (see **fig. 4**). Use the Transform menu first to mark the normal to the mirror and then to reflect the light ray over it. How does a light ray reflect when the incident angle is 0 degrees with respect to the central axis of the parabola? (Note: The GSP sketch models light

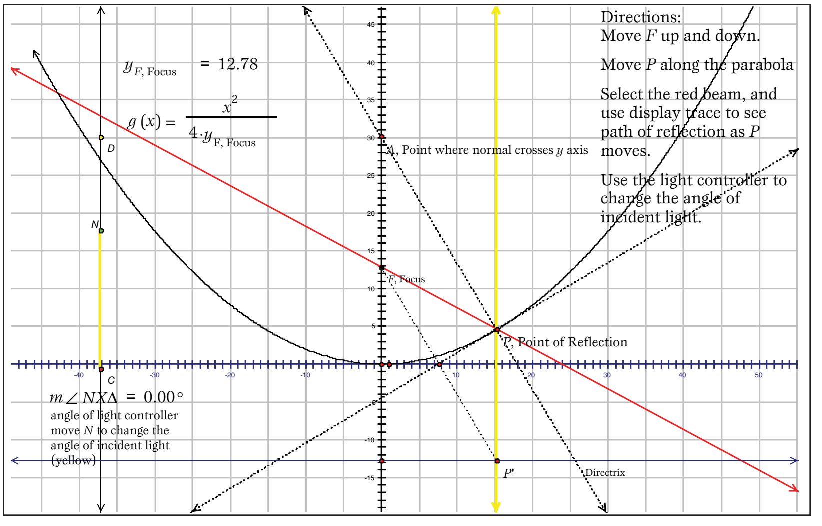


Fig. 4 Incident light and angle control

from a very distant source, such as a star, which is directly in front of a parabolic mirror. For students who are not familiar with GSP, this model can be built as a demonstration.)
4. Show that the geometry of a parabola is consistent with this effect.

Answers
1. For both, the light reflects back across the normal, at the point of reflection.
2. It should reflect about the normal to the parabola.
3. When the light is at 0 degrees, the exiting beam always reflects to the focus.
4. Print copies of or project **figure 4** onto poster paper or a whiteboard. Set the angle of incident light to 0 degrees and locate *P* away from the vertex. By the definition of a parabola, triangle *FPP′* is isosceles, so the measure of angle *PFP′* is equal to the measure of angle *PP′F*. We are certain from investigation 3 that the tangent to the parabola at *P* is perpendicular to the normal and to $\overline{FP'}$. So $\overline{PA}$ and $\overline{FP'}$ are parallel. By noticing this and by using $\overline{P'P}$ and $\overline{FP}$ as transversals, we can use the alternate interior angle theorem and vertical angles to show that the angle made by the vertical line (the light) through *P* and through the normal ($\overline{AP}$) and the angle made through $\overline{FP}$ and through the normal are always equal for all parabolas. The nature of reflection and the structure of the parabola guarantee that if light enters a parabolic mirror parallel to the central axis, for every point of the parabola, it will reflect about the normal to the curve and will always go to the focus.

Investigation 7: Describing parabolic reflection for off-axis parallel light rays
Have students use the GSP file from investigation 6 (see **fig. 4**) to test different angles of incoming parallel light representing off-axis distant light sources, as reflected by mirrors with varying focal lengths. (For this activity, it may be helpful to use the Hide command on *FP′*, the directrix, and the tangent lines.)

Questions
1. With incoming light parallel to the *y*-axis, record the position of the exiting light relative to the focus for three different focal lengths.
2. What happens if the light is at an angle with respect to the central axis?
3. Set your light controller to 5 degrees off the central axis and restrict your points of reflection to one part of the parabola. What do you observe?

Answers
1. These light rays always reflect to the focus, regardless of the focal length selected.
2. As the angle of light increases, parabolic mirrors reflect further from the focus.
3. With incoming light angled at 5 degrees and *P* held close to the vertex, we can easily see that the optical focus moves off the central axis.

MODELING REAL TELESCOPES
In the next four investigations (8–11), the modeling moves from general light reflection to the specifics of the equipment used for the kind of telescopes our students constructed. **Figure 5** is a photograph of the mirror of diameter 6 inches and depression 0.05 inches that we installed in our telescope, which consisted of a sonotube of length 45 inches and diameter 8 inches (available at most home improvement centers). The GSP models that we will use can also be customized to other optical devices, such as ordinary hand mirrors or large research telescopes, such as the Hubble Space Telescope.

Investigation 8: Relating size of telescope mirror to the formula for parabola
Have students apply the formula $y = x^2/4a$, developed from investigation 5, by using the mirror dimensions: 6 inches in diameter with a depression of just 0.05 inches.

Questions
1. Look at the mirror (see **fig. 5**). It looks almost flat. Could a cross-section be a parabola?
2. How can we compare this parabola to the models we have made in GSP?
3. What is the focus of the parabola that models this mirror?

Answers
1. Students may suggest that it is a parabola with a very high focus.
2. By relating coordinates to the dimensions of the mirror, we can calculate a focal length.
3. Using our formula $y = x^2/4\text{a}$, with $x = 3$ inches and $y = 0.05$ inches, we find that the focus is 45 inches.

Fig. 5 Mirror with diameter 6 inches and depression 0.05 inches before installation in class-made telescope

Investigation 9: Changing the model

In this activity, students will refine the GSP model to match the specifications of the telescope mirror and the tube.

Questions

1. How can we make our model in GSP correspond to the real mirror?
2. What happens when the light enters the mirror at an angle?
3. How can we estimate the range of angle entry down the 45-inch tube to the mirror?
4. Use your calculation for angle of entry in the model to find the focal surface.

Answers

1. Set the focus to the point (0, 45) and restrict points of reflection to points on the parabola where $-3 \le x \le 3$ (see **fig. 6**).
2. An optical focus can be found to one side of the mathematical focus.
3. Calculate the most extreme angles of light that could enter the telescope (see **fig. 7**). We have a 45-inch tube of diameter 8 inches. A 6-inch diameter mirror in the center could not collect light at an angle greater than the arctangent of 7/45 inches, or ± 9 degrees.
4. In **figure 6,** a trace on the reflected beam shows 5 optical focal points where incoming light is angled at approximately ± 9, ± 4.5, and 0 degrees. Students may observe that the mirror produces a focal surface that is curved. Note that in practice even this restricted field of view is larger than what can be achieved with these telescopes because of the small size of the second mirror.

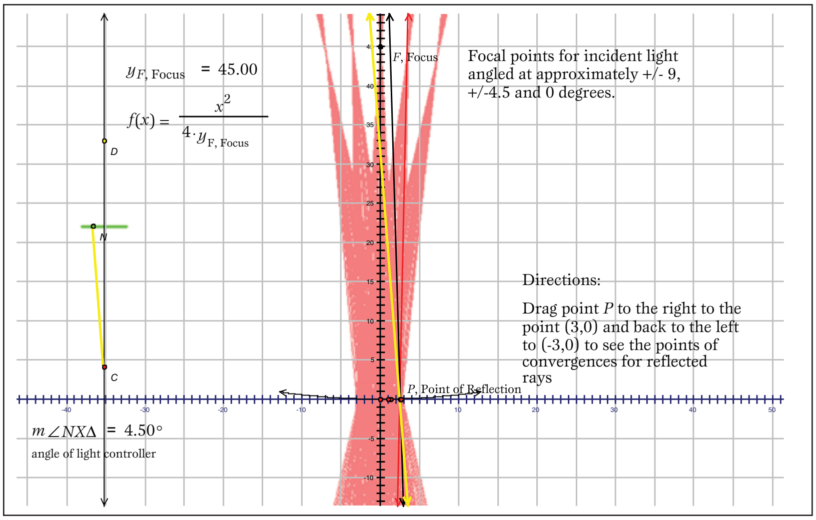


Fig. 6 Points of reflection restricted to $-3 \le x \le 3$

The following investigations should be modified if students have no access to parabolic mirrors, possibly by describing the project activity that followed this part of the field experience. Students used a modified flashlight with cardstock affixed (**fig. 8**) that can project and capture an image in one location to find the point twice the focal length from the vertex.

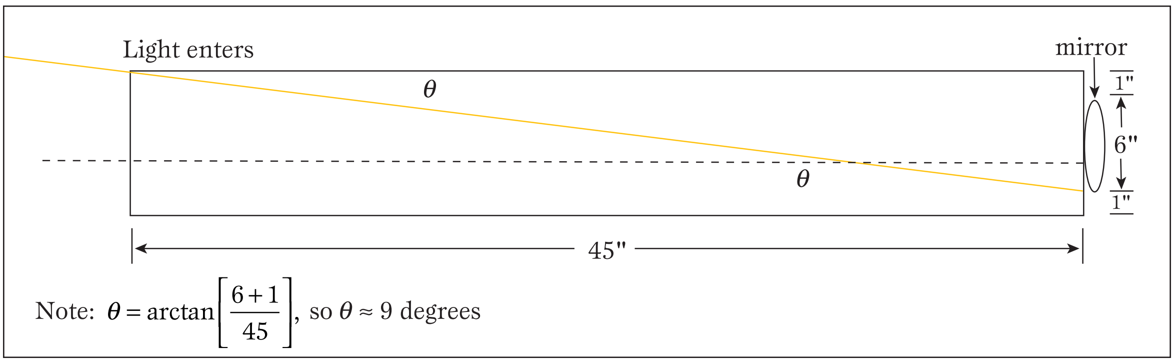


Fig. 7 Partial telescope schematic (eyepiece and second mirror not shown)

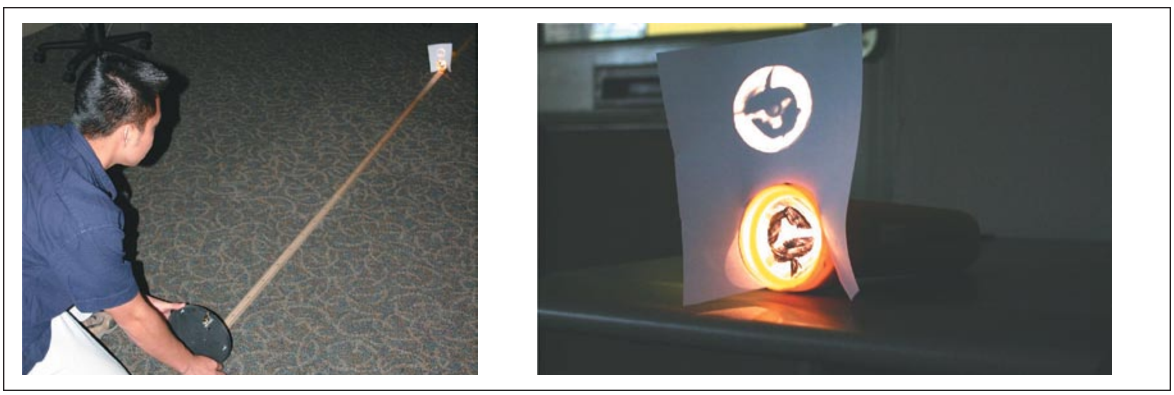


Fig. 8 A student uses a flashlight assembly, telescope mirror, and metersticks to measure twice the focal length.

Investigation 10: Modeling point source reflection

Have students make a GSP model of a parabolic mirror that shows what happens when a nearby light source reflects off the parabolic mirror (see **fig. 9**).

Questions

1. Investigate your new model by moving the point source around. What happens when you sweep the point of reflection around the area close to the vertex of the parabola?
2. Place the point source on the focus. What do you notice?
3. Light reflects about the normal. What happens if we reflect light along the normal?
4. Place a Trace on the axis of reflection. Do the normals converge?
5. Now place the point source where the normals converge. Move *P*. What do you observe? Try this for several parabolas having different focal lengths. For each parabola, use GSP to measure the focal length and the position where the normals cross the *y*-axis.
6. What can we do with a flashlight to find the focus of an unknown mirror?

Answers

1. In **figure 9**, a point source of light is free floating and can be moved anywhere. We can model light that sweeps across the mirror like a fan by moving *P* along the parabola. By restricting *P* to the area close to the vertex to simulate the actual mirror, we can find many optical focus points by changing the location of the point source.
2. Light reflects out in parallel rays aligned with the central axis. This makes sense because light entering a telescope in this way reflects to the focus.
3. Light emitted along the normal should return along the normal to its source.

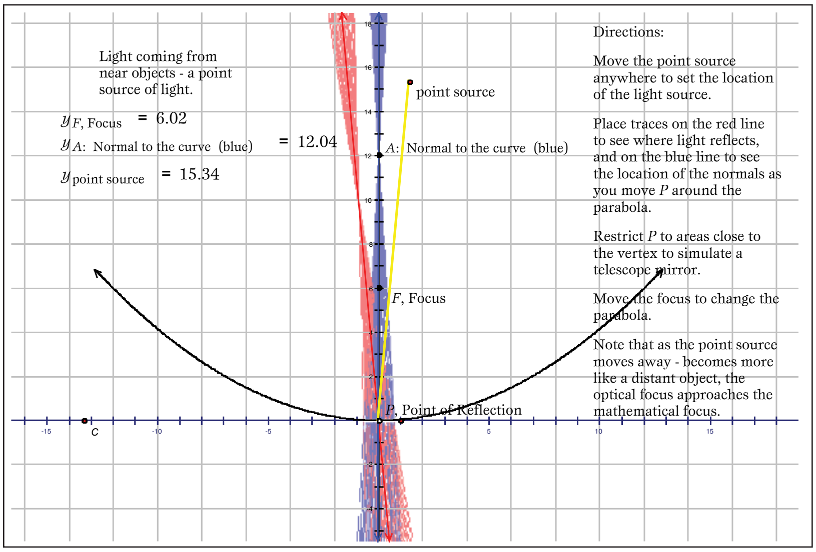


Fig. 9 A GSP model of reflections off a parabolic mirror from a nearby light source

4. The normals appear to converge where they cross the *y*-axis, but only if *P* is restricted to points close to the vertex (see **fig. 9**).
5. The incoming and outgoing light and the normal all coincide. At different focal lengths, the normals still converge at a point twice the height of the focus if *P* is restricted as before.
6. Place a sticker with a simple shape (the letter A, for example) onto the front glass of a flashlight in such a way that the flashlight can still emit light. Also attach a piece of heavy paper or cardstock to the outside of the flashlight, adjacent to the sticker and parallel to the glass, to use as a makeshift screen. By shining the modified flashlight straight into and at varying distances from the mirror, one can find a unique location where an identically sized but inverted image of the sticker's silhouette appears sharp and in focus on the paper screen. As indicated in the previous two questions and further explored in investigation 11, the only location at which the object and the focused image both lie at the same distance from the mirror is twice the mathematical focus of the mirror. Hence, students need only find this location for a mirror, measure the distance to the mirror, and divide by 2 to get the mirror's focal length (see **fig. 8**).

Investigation 11: Demonstrating the geometric properties underlying the relationship between focal length and convergence of normals close to the vertex

In this activity, we show that the geometry of the parabola guarantees the effects just observed. Note that the light source does not affect the geometry of the parabolic mirror. Let *A* be the point where the normal crosses the *y*-axis (see **fig. 4**).

Questions

1. Consider the four-sided figure defined by *A, F, P′* and *P*. What are its properties?
2. What is the *y*-coordinate of *A* when *P* is at (0, 0)?

Answers

1. *F, A, P*, and *P′* form a parallelogram because the normal $\overline{AP}$ is parallel to $\overline{FP'}$ (see investigation 6) and $\overline{AF}$ is parallel to $\overline{PP'}$. Opposite sides of parallelograms are congruent, so $P'P = FA$. By the definition of a parabola, $P'P = PF$. Therefore, $P'P = PF = FA$ for all parabolas.
2. We can find the location of *A* when *P* is close to (0, 0) by using the idea of a collapsing parallelogram. We can see that when *P* is in the neighborhood of (0, 0), then *P′, P, F*, and *A* are nearly collinear with the *y*-axis and in the limit $PF + FA = PA$, or $2PF = PA$. *F* is always at $(0, a)$, so *PF* approaches *a* when *P* approaches (0, 0). This means that the *y* coordinate of *A* will approach $2a$, or twice the focal length,

when P is close to (0, 0). The modified flashlight measurement works because the mirror we are using is very flat with a focus distant in relation to the diameter, bringing us close (actual variation in FA for this mirror is less than 0.2% of the focal length) to the "limit" situation, where PA is exactly 2 times PF.

At the conclusion of investigation 11, algebra 2 students were ready to use flashlight and paper to locate the focus of the mirror manually (see **fig. 8**). During the field experience, we also looked at a GSP model of spherical mirrors and discussed the similarities between spherical and parabolic mirrors. Students can further explore the parabola as a locus of points (Olmstead 1998) or investigate spherical aberration (see the Web site Amazing Space).

CONCLUSION

Our students were very engaged in all aspects of this lesson and enjoyed benefits similar to those reported by Erbas et al. (2005). Connecting paper folding and GSP models as well as introducing the specifics of the mirror and telescope were particularly useful. Students were able to review the material in a variety of ways. By making a model to certain specifications and then exploring the model's properties, students gained a concrete understanding of the properties of a parabola as they apply to a real mirror. They were also challenged to recall and apply their previous knowledge of coordinate systems, the distance formula, reflections, and basic geometric proof. Projecting the GSP models onto the poster paper imparted an interactive element to the whole-group lessons and provided opportunities to draw connections across media.

For the preservice teacher as well as the high school students, this experience was a successful combination of mathematics and astronomy through technology, traditional proof, and project-based collaboration that we hope will be useful to other teachers interested in implementing NCTM's Standards and goals.

REFERENCES

Amazing Space. NASA, Space Telescope Science Institute's Office of Public Outreach. amazing-space.stsci.edu/resources/explorations/groundup/lesson/basics/g14/index.php.

Bogomolny, Alex. Proof. www.cut-the-knot.org/ctk/Parabola.shtml.

Dennis, David, and Jere Confrey. "Functions of a Curve: Leibniz's Original Notion of Functions and Its Meaning for the Parabola." *The College Mathematics Journal* (March 1995): 124–31

Erbas, A. Kursat, Sarah D. Ledford, Chandra Hawley Orrill, and Drew Polly. "Promoting Problem Solving across Geometry and Algebra by Using Technology." *Mathematics Teacher* 98, no. 9 (May 2005): 599–603.

The Geometer's Sketchpad. Emeryville, CA: Key Curriculum Press, 2001. Software.

National Council of Teachers of Mathematics (NCTM). *Principles and Standards for School Mathematics*. Reston, VA: NCTM, 2000.

———. Parabola paper-folding applet. my.nctm.org/eresources/view_article.asp?add=Y&article_id=2074&page=9#.

Olmstead, Eugene A. "Exploring the Locus Definitions of the Conic Sections." *Mathematics Teacher* 91, no. 5 (May 1998): 428–34.

Ryden, Robert. "Astronomical Math." *Mathematics Teacher* 92, no. 9 (December 1999): 786–92.

Sullivan, Michael, and Michael Sullivan III. *Precalculus*. New Jersey: Prentice Hall, 1998. ∞




LAUREN M. SIEGEL, lsiegel@austin.rr.com, has completed the UTeach Program in the College of Natural Sciences, University of Texas, Austin TX 78712, and is currently teaching and developing mathematics curricula at ACE Academy, Austin. GAIL DICKINSON, dickinson@txstate.edu, is an assistant professor in the College of Education, Texas State University, San Marco, TX 78666. She co-designed and taught Project-Based Instruction, the capstone course in the UTeach Natural Sciences Program. ERIC J. HOOPER, ehooper@astro.wisc.edu, worked in the UTeach Program as a National Science Foundation astronomy and astrophysics postdoctoral fellow and currently teaches astronomy and physics at the University of Wisconsin, Madison, WI 53706. MARK DANIELS, mdaniels@math.utexas.edu, has a dual appointment as a clinical associate professor in both the UTeach Program, College of Natural Sciences, and in the mathematics department at the University of Texas, Austin.